JOURNAL OF APPROXIMATION THEORY **45**, 61–68 (1985)

# Best Local Approximation and Generalized Convexity

EITAN LAPIDOT

*44A Eder St., 34752 Haifa, Israel*

*Communicated by Oved Shisha*

Received June 24, 1983; revised July 26, 1984

Best local approximation in sign-monotone norm is discussed. It is shown that if $f \in C^n(I)$, then the (one-sided) best local approximation from an $(n+1)$-dimensional ECT-space exists at every point $x \in I$. If the (two-sided) best local approximation (in the $L_\infty$ or $L_2$-norm) exists and the highest coefficient is positive, then $f$ is $(n-1)$-convex. For more general sign-monotone norms, one is required to assume $n$th order continuous differentiability of the function in order to obtain this result.


## 1. INTRODUCTION

The notion of best local approximation was introduced by Chui, Shisha, and Smith in [4]. They proved that if $\{u_0, u_1, ..., u_{n-1}\}$ is a $T$-system on $[a, b]$ with $W(u_0, u_1, ..., u_{n-1}; a) = \det(u_i^{(j)}(a))_{i,j=0}^{n-1} \neq 0$, then the net $\{T_{n-1}(f; [a, a+\varepsilon])\}$ of best uniform approximations to $f \in C^{n-1}[a, b]$, from $\Lambda_{n-1} = \text{span}\{u_0, u_1, ..., u_{n-1}\}$ converges as $\varepsilon \to 0^+$. The limit function, $T_{n-1}^a f$ is the element of $\Lambda_{n-1}$ that satisfies $(T_{n-1}^a f)^{(j)}(a) = f^{(j)}(a)$, $j = 0, 1, ..., n-1$.

Later, Chui, Smith, and Ward showed the same result for best $L_2$-approximations [5]. Recently, Wolfe [12] generalized this result to any $L_p$-norm, $1 \leqslant p \leqslant \infty$. The convergence of the best approximations on $[a, a+\varepsilon]$ to $T_{n-1}^a f$ is uniform.

In this note we show that the same result holds for a more general family of norms, namely, the sign-monotone norms. Also, we characterize the generalized convexity with respect to $\{u_0, u_1, ..., u_{n-1}\}$ of a function $f$, by its best local approximations.

## 2. BEST LOCAL APPROXIMATION OF A CONVEX FUNCTION

We start by recalling some definitions and results that will be used in the sequel.



0021-9045/85 $3.00

The set of functions $\{u_0, u_1,..., u_k\}$ is called a Tchebycheff system ($T$-system) on $[a, b]$ if

$$U\begin{pmatrix} u_0, u_1,..., u_k \\ t_0, t_1,..., t_k \end{pmatrix} = \det(u_i(t_j))_{i,j=0}^k > 0 \qquad (1)$$

whenever $a \leqslant t_0 < t_1 < \cdots < t_k \leqslant b$.

The system $\{u_0, u_1,..., u_n\}$ is called an Extended-Complete $T$-system (ECT-system) if (i) Equation (1) holds for every $k$, $k = 0, 1, 2,..., n$, and (ii) equalities may occur among the $t_i$'s. In this case the appropriate columns are replaced by successive derivatives (see, e.g., [6, p. 6]). The $u_i$'s are assumed to be in the continuity class $C^n[a, b]$. With no loss of generality we may assume that

$$u_i(t) = \phi_i(t; a), \qquad i = 0, 1, 2,..., n, \qquad (2)$$

where

$$\begin{aligned} \phi_i(t; x) &= w_0(t) \int_x^t w_1(\xi_1)... \int_x^{\xi_{i-1}} w_i(\xi_i)\, d\xi_i ... d\xi_1, \qquad & x \leqslant t \leqslant b, \\ &= 0, & a \leqslant t < x, \end{aligned} \qquad (3)$$

and where $w_k \in C^{n-k}[a, b]$ and is positive for every $k = 0, 1, 2,..., n$.

A function $f \in C(a, b)$ is said to be $k$-convex (with respect to the $T$-system $\{u_0, u_1,..., u_k\}$) if

$$U\begin{pmatrix} u_0, u_1,..., u_k, f \\ t_0, t_1,..., t_k, t_{k+1} \end{pmatrix} \geqslant 0 \qquad (4)$$

for all $a < t_0 < t_1 < \cdots t_{k+1} < b$.

The set of all $k$-convex functions with respect to the system $\{u_0, u_1,..., u_k\}$ is a convex cone denoted by $C(u_0, u_1,..., u_k)$.

A function $f$ is said to be $k$-convex on a subinterval $I$ of $[a, b]$, if (4) holds whenever the $t_i$'s are in $I$. $f$ is said to be $k$-concave on $I$ if $-f$ is $k$-convex.

For $i = 0, 1, 2,..., n$, let $D_i = (d/dt)(\cdot/w_i(t))$ be a first order differential operator and let $D_{-1} f = f$. Also we set $D^i = D_i D^{i-1}$, where $D^{-1} = D_{-1}$. If $f \in C^n(a, b)$ then it admits the Taylor type formula

$$f(t) \sim \sum_{i=0}^{n} ((D^{i-1} f)/w_i)(x)\, \phi_i(t; x), \qquad t \in [x, b), \qquad (5)$$

for every $x \in (a, b)$. If $f$ has a right-hand side $n$th derivative at $x = a$, then (5) holds with $x = a$.

In [8] sign-monotone norms are defined on $C[a, b]$. A norm $\|\cdot\|$ is said to be sign-monotone if $f(x)\cdot g(x)\geq 0$ and $|f(x)|\geq|g(x)|$ for all $x\in[a, b]$ imply $\|f\|\geq\|g\|$. For every subinterval $I=[\alpha, \beta]$ of $[a, b]$, a sign-monotone seminorm $\|\cdot\|_I$ is defined by $\|f\|_I=\|f\circ\phi_I\|$, where $\phi_I(t)=((\beta-\alpha)\,t+\alpha b-\beta a)/(b-a)$.

We denote by $\Lambda_k$ the span of $\{u_0, u_1,..., u_k\}$. The elements of $\Lambda_k$ are called $\Lambda_k$-polynomials. A function $f$ is called a nowhere $\Lambda_k$-polynomial if there does not exist an interval $[\alpha, \beta]\subset[a, b]$ on which it agrees with some $u\in\Lambda_k$.

Let $f\in C[a, b]$. $T_k(f; I)$ denotes a best approximation to $f$ from $\Lambda_k$ in $\|\cdot\|_I$. (In case there is more than one best approximation, $T_k$ is chosen arbitrarily to be any one of them.)

Finally if the net $\{T_n(f; [x, x+\varepsilon])\}$ converges as $\varepsilon\to 0^+$ then the limit

$$T_n^x f=\lim_{\varepsilon\to 0^+} T_n(f; [x, x+\varepsilon]) \tag{6}$$

is called the (right-hand side) best local approximation to $f$ at $x$. We show that although $T_n(f; [x, x+\varepsilon])$ are not necessarily unique, $T_n^x f$ is unique.

Since $W(u_0, u_1,..., u_n)>0$ (see [6, Theorem 1.2, p. 379]) we can prove the following:

THEOREM 1. *Let* $\{u_i\}_{i=0}^n$ *be an* ECT-*system on* $[a, b]$ *having the representation* (2) *and* (3) *and let* $f$ *be a nowhere polynomial element of* $C^n[a, b]\cap C(u_0, u_1,..., u_{n-1})$. *Then for every* $x\in[a, b]$, *the best local approximation to* $f$ *from* $\Lambda_n$ *exists and* $T_n^x f(t)=\sum_{i=0}^n ((D^{i-1}f)/w_i)(x)\,\phi_i(t; x)$ *for* $t\in[x, b]$. *In particular if* $T_n^x f=\sum_{i=0}^n a_i u_i$ *then* $a_n>0$.

The proof follows along lines similar to those of [4, Theorem 2.1].

## 3. CONVERSE THEOREMS

A converse theorem does not hold. For, consider the following

EXAMPLE 1. Let

$$\begin{aligned} f(t) &= t^{n-1}(t-1), \qquad -1\leq t<0,\\ &= t^{n-1}(t+1), \qquad 0\leq t<1. \end{aligned}$$

$T_n^x f(t)$ is either $t^n-t^{n-1}$ or $t^n+t^{n-1}$, $a_n>0$, for every $x$, however, $f\notin C(1, t,..., t^{n-1})$ on $(-1, 1)$.

In order to prove a converse theorem we have to confine ourselves to two-sided best local approximations.

DEFINITION. Let $f$ be defined on a subinterval of $[a, b]$ containing $x$. If the two limits, $\lim_{y \to x^-} T_n(f; [x, y])$ and $\lim_{z \to x^-} T_n(f; [z, x])$, exist and are equal to each other, then the common limit is called the best local approximation of $f$ at $x$ and is denoted by $T_n^x f$.

We divide our work into two parts. First we prove the converse theorem for the uniform and the $L_2$ norms and then for general sign-monotone ones. For the sake of simplicity we introduce the following notation: if $u = \sum_{i=0}^n a_i u_i$ then $a_i(u) = a_i$.

LEMMA 1. *Let $f$ be a nowhere $A_{n-1}$-polynomial, continuous function on $(a, b)$. If $f \notin C(u_0, u_1,..., u_{n-1})$, then there exists a point $a < x < b$, such that each of its neighbourhoods contains an interval $[\alpha_x, \beta_x]$, containing $x$, with $a_n(T_n(f; [a_x, \beta_x])) < 0$.*

*Proof.* If $f \notin C(u_0, u_1,..., u_{n-1})$ then there exists an interval $[\alpha, \beta] \subset (a, b)$ such that

$$a_n(T_n(f; [\alpha, \beta]) < 0. \tag{7}$$

(See [1] for the uniform norm and [2] for the $L_2$-norm.)

We now show that $[\alpha, \beta]$ contains a subinterval $[\alpha', \beta']$ with $\beta' - \alpha' \leqslant (\beta - \alpha)/2$ such that

$$a_n(T_n(f; [\alpha', \beta'])) < 0. \tag{8}$$

Assume to the contrary that no such subinterval exists. In particular, no such interval is contained in $[\alpha, (\alpha + \beta)/2]$, $[(\alpha + \beta)/2, \beta]$, or $[(\alpha + \beta)/4, 3((\alpha + \beta)/4]$. Hence,

$$U\begin{pmatrix} u_0, u_1,..., u_{n-1}, f \\ t_0, t_1,..., t_{n-1}, t_n \end{pmatrix} \geqslant 0 \tag{9}$$

whenever $t_0 < t_1 < \cdots < t_n$ are $n + 1$ points in any of these intervals. Since $f$ is a nowhere $A_{n-1}$-polynomial, all the determinants (9) are strictly positive [9], i.e., $\{u_0, u_1,..., u_{n-1}, f\}$ is a T-system on each of the three intervals and by [10] it is a T-system on $[\alpha, \beta]$ in contradiction to (7), which completes the proof of the lemma.

Let $\varepsilon > 0$ be given. There exists a number $\delta = \delta(\varepsilon)$ such that for every $y$, $y \in (x, x + \delta)$, $\|T_n^x f - T_n(f; [x, y])\| < \varepsilon/2$. For every $y \in [x, x + \delta)$, set $L_y = \{t \mid t \in [a, x] \cap (x - \delta, x]$ such that $\|T_n^x f - T_n(f; [z, y]))\| < \varepsilon$ for every $z$, $t < z < x\}$. Obviously $L_y$ is an interval. Now let $l(y) = \inf L_y$. Clearly $l(y) < x$ for every $y \in [x, x + \delta)$.

We now show:

LEMMA 2. *For every* $y \in [x, x+\delta)$, $\limsup_{y \to y_0} l(y) \leqslant l(y_0)$, *i.e.*, $l(y)$ *is upper semicontinuous.*

*Proof.* Assume that there exists a sequence $\{y_i\}$, $y_i \in (x, x+\delta)$, with $\lim_{i \to \infty} y_i = y_0$ such that $\lim_{i \to \infty} l(y_i) = t_0 > l(y_0)$. By the definition of $l(y)$, $\|T_n^x f - T_n(f; [l(y_i), y_i])\| = \varepsilon$ (or else $l(y_i) = a$) and from the continuity of $T_n$ (in the interval) one concludes that $\|T_n^x f - T_n(f; [t_0, y_0])\| = \varepsilon$ (or else $t_0 = a$ in which case $t_0 \leqslant l(y_0)$), which contradicts the definition of $l(y_0)$.

Similarly we set for $z \in (x-\delta, x]$, $V_z = \{t \mid \in [x, b] \cap [x, x+\delta)$ such that $\|T_n^x f - T_n(f; [x, y])\| < \varepsilon$ for every $y$, $x < y < t\}$ and $v(z) = \sup V_z$. One can show that $\liminf_{z \to z_0} v(z) \geqslant v(z_0)$ for every $z \in (x-\delta, x]$, i.e., $v(z)$ is lower semicontinuous.

LEMMA 3. *Let* $x \in (a, b)$; *if* $T_n^x f \in C(u_0, u_1, ..., u_{n-1}) \backslash A_{n-1}$ *then there exists an interval* $[l, v] \subset (a, b)$, *containing* $x$, *such that for every interval* $[\alpha, \beta]$ *with* $l < \alpha \leqslant x \leqslant \beta < v$, $T_n(f; [\alpha, \beta]) \in C(u_0, u_1, ..., u_{n-1}) \backslash A_{n-1}$.

*Proof.* Let $l = \sup\{l(y) \mid y \in [x, x+\delta)\}$ and let $v = \inf\{v(z) \mid z \in (x-\delta, x]\}$. By Lemma 2, $l < x < v$ and if $\varepsilon$ is sufficiently small $[l, v]$ has the desired property. (Note that if $\varepsilon$ is sufficiently small then $\|T_n^x f - T_n(f; [\alpha, \beta])\| < \varepsilon$ implies that $T_n(f; [\alpha, \beta]) \in C(u_0, u_1, ..., u_{n-1}) \backslash A_{n-1}$.)

THEOREM 2. *Let* $f \in C(a, b)$. *If for each* $x \in (a, b)$, $a_n(T_n^x f) > 0$ *then* $f \in C(u_0, u_1, ..., u_{n-1}) \backslash A_{n-1}$.

*Proof.* First note that $f$ is a nowhere $A_{n-1}$-polynomial. If $f \notin C(u_0, u_1, ..., u_{n-1})$ then by Lemma 1, $a_n(T_n(f; [\alpha_x, \beta_x])) < 0$ for arbitrarily small intervals containing $x$, which contradicts Lemma 3.

The following example shows that the conditions in Theorem 2 are not necessary.

EXAMPLE 2. Let

$$f(t) = t^n, \qquad -1 < t < 0,$$
$$= t^n + t^{n-1}, \qquad 0 \leqslant t < 1.$$

$f$ is a nowhere $A_{n-1}$-polynomial, $f \in C(1, t, ..., t^{n-1})$, but $T_n^0 f$ does not exist.

## 4. Best Approximation from a Lower Dimensional T-Space

In the previous section we discussed the relations between the best local approximation from the $(n+1)$-dimensional $T$-space $A_n$ and its $(n-1)$-convexity. Since one does not need $u_n$ in order to define the above mentioned convexity, one may ask whether it is possible to characterize this convexity by best approximations from $A_{n-1}$. In this section we answer the question in the affirmative. Moreover, as follows by Theorem 3 below, best local approximation is a most natural characterization of $(n-1)$-convexity.

We first show that some $(n-2)$-convexity properties are equivalent to the $(n-1)$st one.

LEMMA 4. *Let $\{u_i\}_{i=0}^{n-1}$ be an* ECT-*system on $[a, b]$ and let $f$ be defined on $(a, b)$. $f \in C(u_0, u_1, \ldots, u_{n-1})$ iff for every $x \in (a, b)$ there exists an element $u_x \in A_{n-1}$ such that $f - u_x$ is $(n-2)$-concave on $(a, x]$ and $(n-2)$-convex on $[x, b)$. (Note that $(-1)$-convexity means positivity.)*

*Proof.* The necessity is clear since $D_R^{n-1} f$ increases [6, p. 386].

*Sufficiency.* $n=1$. We may assume that $u_0 = 1$. Let $t_1 < t_2$. Set $x = (t_1 + t_2)/2$, $(f - u_x)(t_1) < 0$, and $(f - u_x)(t_2) > 0$, hence $f(t_1) < f(t_2)$.

$n = 2$. Assume as before that $u_1 = 1$. $f$ will be convex with respect to $\{1, u_1\}$ on an interval $I$ iff $f \circ u_1^{-1}$ is convex with respect to $\{1, t\}$ on $u_1(I)$. Thus we may assume that $u_1(t) = t$. Let $t_1 < t_2 < t_3$. $(f - u_{t_2})(t_1) > (f - u_{t_2})(t_2)$ and $(f - u_{t_2})(t_3) > (f - u_{t_2})(t_2)$. If $t_2 = \alpha t_1 + (1-\alpha) t_3$, where $0 < \alpha < 1$, then $\alpha f(t_1) + (1-\alpha) f(t_3) > f(t_2)$.

$n \geq 3$. It is known that $f \in C(u_0, u_1, \ldots, u_{n-1})$ iff $D_0 f \in C(D_0 u_1, D_0 u_2, \ldots, D_0 u_{n-1})$. The proof proceeds by induction.

THEOREM 3. *Let $\{u_i\}_{i=0}^{n-1}$ be an* ECT-*system on $[a, b]$ and let $f \in C(a, b)$. If $T_{n-1}^x f$ exists for every $x$ and $a_{n-1}(T_{n-1}^x f)$ strictly increases with $x$ then $f \in C(u_0, u_1, \ldots, u_{n-1})$.*

*Proof.* For every $x \in (a, b)$, Theorem 2 implies that $f - T_{n-1}^x f$ is $(n-2)$-concave on $(a, x]$ and $(n-2)$-convex on $[x, b)$. Thus, by Lemma 4, $f \in C(u_0, u_1, \ldots, u_{n-1})$.

## 5. The Characterization Theorem for Sign-Monotone Norms

The characterization of the convexity of $f$ by means of its best approximations in a general sign-monotone norm requires an additional assumption on $f$, namely, $f \in C^n(a, b)$. Under this assumption Amir and Ziegler [3] proved that if $a_n(T_n(f; I)) \geq 0$ for every $I \subset (a, b)$, where

$T_n(f; I)$ is the best $L_p$-approximation to $f$ on $I$, then $f \in C(u_0, u_1, ..., u_{n-1})$. Kimchi generalized this result to (continuous) sign-monotone norms (see [7, Theorem 3.2]). Moreover, if $T_n f$ is a best approximation to $f$ from $A_n$ in any sign-monotone norm and if $f$ is a nowhere $A_n$-polynomial then the number of zeros of $f - T_n f \geq n+1$, the zeros are counted up to "multiplicity" 2 (see [7, 8]). Thus under the differentiability assumption on $f$ we can prove the following:

THEOREM 2′. *Let $f \in C^n(a, b)$ and let $T_{n,}^x f$ denote its best local approximation at $x$ in a sign-monotone norm. If for every $x \in (a, b)$, $a_n(T_{n,}^x f) > 0$ then $f \in C(u_0, u_1, ..., u_{n-1}) \backslash A_{n-1}$.*

We also have:

THEOREM 3′. *Let $f \in C^n(a, b)$ and let $T_{n-1,}^x f$ be its best local approximation at $x$, in a sign-monotone norm. If $a_{n-1}(T_{n-1,}^x f)$ strictly increases with $x$ then $f \in C(u_0, u_1, ..., u_{n-1}) \backslash A_{n-1}$.*

The following examples show that the existence of best local approximation from any T-space to a function $f$ does not imply the differentiability of $f$. This implies that Theorems 2 and 3 cannot hold for a general sign-monotone norm, and the additional assumptions of Theorems 2′ and 3′ are required.

EXAMPLE 3. Let $f(t) = \sin(1/t)$ for $t \in [-1, 0) \cup (0, 1]$ and $f(0) = 0$. For every T-space $A_n$, $T_{n,}^0 f = 0$ (in the $L_\infty$-norm) although $f$ is not even continuous on $[-1, 1]$.

EXAMPLE 4. Let $f \in C[-1, 1]$ be defined as follows: for $t \in E_0 = [0, 1] \backslash \bigcup_{k=0}^{\infty} (1-\varepsilon)/2^k, 1/2^k)$ $(0 < \varepsilon < \frac{1}{2})$, set $f(t) = 0$. For $k = 0, 1, 2, ...$ set $f((1 - 2^{-1}\varepsilon)/2^k) = (1 - 2^{-1}\varepsilon)/2^k$, and then define $f$ to be linear on each of the two closed halves of $[(1-\varepsilon)/2^k, 1/2^k]$, $k = 0, 1, 2, ...$.

Finally, let $f(t) = -f(-t)$ for $t \in [-1, 0)$. Given a T-space $A_n$, on $[-1, 1]$, $\varepsilon$ can be chosen sufficiently small that the best local $L_1$-approximation $T_{n,}^0 f = 0$ (see [11]). However, $\liminf_{t \to 0} (f(t)/t) = 0$ and $\limsup_{t \to 0} (f(t)/t) = 1$; i.e., $f'(0)$ does not exist.

## REFERENCES


1. D. AMIR AND Z. ZIEGLER, Functions with strictly decreasing distances from increasing Tchebycheff subspaces, *J. Approx. Theory* **6** (1972), 332–344.
2. D. AMIR AND Z. ZIEGLER, Characterization of generalized convex functions by best $L^2$-approximations, *J. Approx. Theory* **14** (1975), 115–127.

3. D. AMIR AND Z. ZIEGLER, Characterization of Generalized Convex Functions by their Best $L^p$-Approximations, *in* "Approximation Theory" (G. G. Lorentz, Ed.), Academic Press, New York/London, 1973.
4. C. K. CHUI, O. SHISHA, AND P. W. SMITH, Best local approximation, *J. Approx. Theory* **15** (1975), 371–381.
5. C. K. CHUI, P. W. SMITH, AND J. D. WARD, Best $L_2$ local approximation, *J. Approx. Theory* **22** (1978), 254–261.
6. S. KARLIN AND W. J. STUDDEN, Tchebycheff systems: With applications in analysis and statistics, *in* "Pure and Applied Mathematics," Vol. XV, Wiley, New York, 1966.
7. E. KIMCHI, Characterization of generalized convex functions by their best approximation in sign monotone norms, *J. Approx. Theory* **24** (1978), 350–360.
8. E. KIMCHI AND N. RICHTER-DYN, A necessary condition for best approximation in monotone and sign-monotone norms, *J. Approx. Theory* **25** (1979), 169–175.
9. E. LAPIDOT, On complete Tchebycheff-systems, *J. Approx. Theory* **23** (1978), 324–331.
10. E. LAPIDOT, On generalized mid-point convexity, *Rocky Mountain J. Math.* **11** (1981), 571–575.
11. J. ROSENBLATT, Determining sets and best $L_1$-approximation, *J. Approx. Theory* **32** (1981), 103–114.
12. J. M. WOLFE, Interpolation and best $L_p$ local approximation, *J. Approx. Theory* **32** (1981), 96–102.